Separation Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713708471

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To cite this Article Olivier, J. , Vaxelaire, J. and Vorobiev, E.(2007) 'Modelling of Cake Filtration: An Overview', Separation Science and Technology, 42: 8, 1667 — 1700

To link to this Article: DOI: 10.1080/01496390701242186

URL: http://dx.doi.org/10.1080/01496390701242186

Separation Science and Technology, 42: 1667–1700, 2007

ISSN 0149-6395 print/1520-5754 online
DOI: 10.1080/01496390701242186

Modelling of Cake Filtration: An Overview

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Abstract: This work presents an overview of literature relative to the modelling of cake filtration and expression. Different approaches are compared to the so called conventional one in order to underline particular analogies and differences. In this way the derivation of fundamental equations (continuity equation and momentum balance) and the determination of the constitutive equations are discussed. Application of current models to complex materials such as sludge and pulp, and to simulation and design of industrial devices is also discussed.

Keywords: Consolidation, dewatering, expression, filtration

INTRODUCTION

The processes of solid liquid separation by filtration and/or expression are widely used in many industrial sectors ranging from chemical, pharmaceutical, agricultural, and mineral production to environmental protection. Despite being based on a process that appears relatively simple, the design and the verification of the installations remain largely empirical, and progress in terms of modelling needs to be made.

Received 28 November 2006, Accepted 14 January 2007

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Ruth et al.'s work (1, 2) is often cited as being pioneering in this field. However, several earlier studies exist which enabled the development of certain fundamentals of filtration theory (3–7). Ruth et al.'s approach is based on analogy with Ohm's Law in which the total resistance is the summation of the filter medium resistance and that of the cake:

$$\frac{dV_f}{dt} = \left(\frac{\Delta P_T\ \Omega^2}{\mu_1\, \alpha_{v,av}\ \nu}\right)\frac{1}{V_f} + \frac{\Delta P_T\ \Omega}{\mu_l\, R_{fm}} \tag{1}$$

With ΔP_T the total pressure drop across the whole system (cake + filter medium), V_f the volume of filtrate, μ_1 the liquid viscosity, Ω the cross-sectional area of filtration, v the ratio between the volume of cake deposited and the volume of filtrate collected, $\alpha_{v,av}$ the cake average specific resistance in volume expressed in m^{-2}, and R_{fm} the resistance of the filter medium (m^{-1}).

For rigid filter cakes v is constant and the integration of Eq. 1 in the case of filtration at constant pressure results in:

$$\frac{t}{V_f} = \frac{\mu_l\, \alpha_{v,av}\ \nu}{2\Delta P_T\, \Omega^2} V_f + \frac{\mu_l\, R_{fm}}{\Delta P_T\, \Omega} \tag{2}$$

This last equation is widely used in industry to design processes and also for filtration research. In many studies the specific resistance in mass is preferred, the product $\alpha_{v,av}\ v$ is then replaced by the product $\alpha_{m,av}\ W$ with W the mass of the solid deposited by volume of filtrate collected (8, 9). In particular, Eq. 2 enables, from experiments carried out at constant pressure and represented within the coordinates t/V_f against V_f, the calculation of $\alpha_{v,av}$ or $\alpha_{m,av}$ (from the slope) and R_{fm} (the ordinate at the origin). This type of representation also identifies the phases of filtration and expression (Fig. 1). The filtration is characterized by the linear part. It corresponds to the formation of a cake due to the accumulation of the solid particles on the surface of a filter medium. The second section represents the expression. It describes the

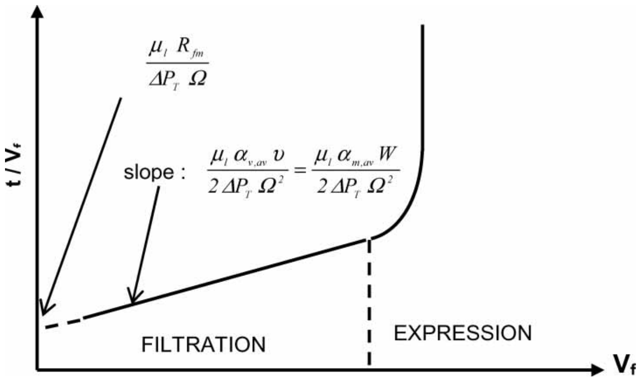

Figure 1. Schematic curve of filtration-compression.

removal of water by cake squeezing. This expression phase is important in order to take into account highly compressible materials such as pulps or sludge. It especially occurs in devices such as the diaphragm filter press or the belt filter press. The difference between filtration and expression phases can be well-illustrated by a sketch showing suspension dehydration in Filtration Compression Cell (Fig. 2).

Although Eqs. (1) and (2) are good models for certain materials (known as incompressible or barely compressible) e.g. talc, it is highly debatable, whether it works, for those materials for which the hypothesis of a rigid cake is not acceptable e.g. waste biological sludge. Furthermore, these equations do not describe the cake structure (in particular in terms of porosity) during its formation, consolidation, and expression. Ever since the first work was carried out by Carman (7) on these aspects, numerous studies have been done, and are still being undertaken, in order to offer less empirical and more detailed models of the process.

Researchers from different scientific backgrounds (essentially chemical engineering, soil mechanics, environmental, and civil engineering) have all entered this field often with different approaches and a lack of coherence as far as notation is concerned is blatant. This renders comparison difficult. According to Tien (10) "Studies aimed at reconciling these results and their comparison (in terms of user-friendliness and accuracy) are therefore needed." Only a few review papers have been published on this subject. The well-known Wakeman's paper (11) was published in 1981. The Shirato's and co-workers paper (12) is dated from 1985. More recent reviews by Lee and Wang (13) and Yükseler et al. (14) allow stock to be taken of the work undertaken in the field of filtration and expression; however, on several points their analysis remains insufficient which in our opinion justifies the presentation of a complementary study.

The aim of this article is therefore a comparative analysis of the different models and approaches whilst highlighting, to a maximum, their particular

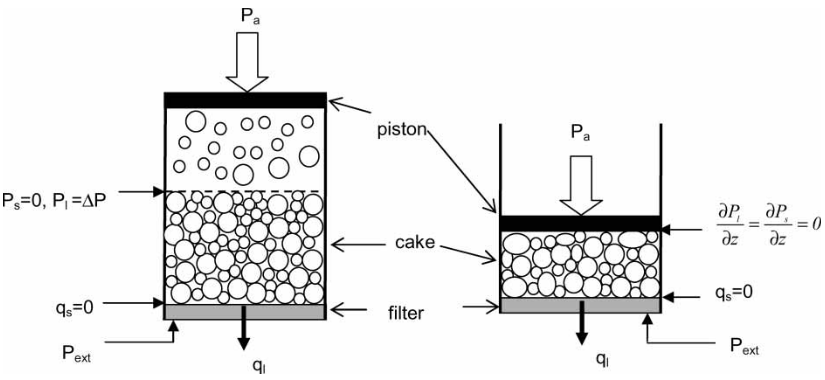

Figure 2. One dimensional filtration and expression processes.

analogies and differences. In order to render the reading of the different equations user-friendly the more conventional approach is taken as a reference and the other models features are discuss according to this approach.

MAIN FEATURES OF THE DIFFERENT MODELLING APPROACHES

The different filtration models are all derived on the same basis, fundamental equations (continuity equation and momentum balance) and material properties parameters (compressibility and permeability). The discrepancies between the different approaches are essentially due to the way these equations and parameters are derived.

The conventional modelling, widely developed by Tiller and co-workers (15–18), used a Darcian approach. It is based on two key parameters, the specific cake resistance (α), and the solid pressure (P_S), to describe the liquid flow through the cake and the consolidation stages, respectively. The notion of specific cake resistance comes from pioneer works on filtration mechanics, such as the ones of Ruth et al. (1, 2) whereas the origins of the solid pressure concept mostly refer to soil mechanics field. The solid pressure (also called "compressive stress" (16, 19–22), compressive drag pressure (17), cumulative stress or accumulative drag pressure (18, 23), contact pressure (24), structure stress (25) represents a core concept in conventional filtration modelling. It considers that a force (F_S) due to the flow of the liquid through the filter cake is transmitted by friction with the solid particles. This force transmitted by the liquid to the solid builds up from particle to particle, up to a maximum at the filter medium level. At the same time, the liquid loses some of its energy and its pressure decreases to reach its lowest at the filter medium level.

The approach emerging from Buscall and White works (26), and initially supported by Landman et al. (27–30) in the filtration field, is relatively close to the conventional one in terms of framework. The difference is rather conceptual (31). Indeed, this modelling considers the quantities which define liquid flow through the cake and consolidation stages more as "material properties" than as parameter dependant of process application. In this way, the material is considered as a network possessing a compressive yield stress (P_y) (compressive rheology concept). This stress is assumed to be an implicit function of the bonding forces between the particles and of "previous shear history of the system during the flocculation process," and an explicit function of the local volume fraction of the solid phase. It only appears in the system above a certain volume fraction ϕ_g (gel point) which corresponds to a system which is fully networked. Below this volume fraction, $P_y = 0$. Furthermore, Landman et al. (27–29) assumed that during filtration, when the stress applied on the solid (P_s) exceeded the pressure P_y the network (cake) reorganized itself instantaneously to form a structure in which the porosity permitted

$P_y = P_s$. Otherwise, the permeability of the cake is regarded on the basis of a settling Stokes' law via a "hindered settling factor," r(ϕ), which characterize hydrodynamic drag and network consolidation.

The modeling proposed by Smiles et al. (32–34) is quite different to the two approaches presented above. It used the "piezometric potential" of liquid (which can be directly measured by using a piezometric tube) instead of liquid pore pressure in the Darcy's law and derives the filtration equation in the form of a diffusion-like equation. This formulation is not significantly used through the filtration literature, apparently because its development is conceptually difficult (35, 36).

As another way of modeling, Willis and co-workers (37–40) based their works on the multiphase theory. They derived the fundamental Equations in 3D at the scale of each of the phases (each one assumed to be continuous). Then to overcome the poor knowledge of cake structure they carried out a change of scale by volume averaging (41, 42). The link between the relations established on each phase is ensured by the usual boundary conditions holding for continuous media (velocity and stress continuities). This averaging technique consists in integrating the equations written at the phase scale, on a characteristic averaging volume for which all physical quantities are defined by their average on the averaging volume. The change over to this new scale called local (or macroscopic) allows the averaging volume to be considered as a material point of an equivalent homogenous medium. Although it was considered by Willis and Tosun (37) as a more rigorous derivation, this modelling is used little through the literature; the conventional approach and the model based on the Buscall and White works (26) are the most used currently (43).

In the majority of works relative to filtration modelling sedimentation is neglected. The reader interested in the taking into account of this phenomenon should consult the references (44–46).

FUNDAMENTAL EQUATIONS

Continuity Equations

The continuity equations are based on several usual hypotheses used in the field of solid-liquid separation by filtration, namely: the two phases (solid and liquid) do not react together and are insoluble; the process is isothermal and the two phases are incompressible (their volume remains constant). In the conventional approach, the continuity equation on the liquid was derived from a one-dimensional mass balance, on a cake layer of thickness dz (47). Adopting the direction of the z-axis from the filter medium to the top surface of the cake, it becomes:

$$\frac{\partial \dot{m}_l}{\partial z} dz = \frac{\partial m_l}{\partial t} \tag{3}$$

with m_l and $\dot{m}_l$ the liquid mass and mass flow rate, respectively. By introducing the porosity (ε), the interstitial velocity of the liquid ($v_l = \dot{m}_l/(\rho_l)\ \varepsilon\Omega$) and superficial flow rate ($q_1 = \varepsilon v_1$), the Eq. (3) can be rewritten using the form:

$$\frac{\partial q_l}{\partial z} = \frac{\partial \varepsilon}{\partial t} \tag{4}$$

The same type of balance, derived on the solid phase, enables the formulation of an equation relating the superficial flow rate of solid ($q_s = \phi\ v_s$) and the solid volume fraction (ϕ):

$$\frac{\partial q_s}{\partial z} = \frac{\partial \phi}{\partial t} \tag{5}$$

These mass balances derived on both phases can be coupled together taking note that $\phi + \varepsilon = 1$. Therefore by addition it becomes:

$$\frac{\partial q_s}{\partial z} + \frac{\partial q_l}{\partial z} = 0 \tag{6}$$

On the basis of the multiphase approach the continuity equations can be expressed after averaging as:

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \bar{\vec{v}}_l^{\,l}) = 0 \tag{7}$$

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \bar{\vec{v}}_s^{\,s}) = 0 \tag{8}$$

with $\bar{\vec{v}}_l^{\,l}$ and $\bar{\vec{v}}_s^{\,s}$ the intrinsic average velocity[1] of the liquid-phase and solid-phase, respectively.

Eqs. (7) and (8) are comparable, in one dimension, to the equations established by the conventional approach Eqs. (4) and (5), by identifying $\bar{\vec{v}}_l^{\,l}$ to v_1 and $\bar{\vec{v}}_s^{\,s}$ to v_s.

In literature relative to filtration modelling, there is little discussion concerning mass balance because of the similarity found in the final equations (whatever the approach). Nevertheless, one can notice that for the modelling of biological press-cake compression Lanoisellé et al. (48) modified the continuity Eq. (4) to better represent the cake which consists, in this particular case, of a layer of particles containing liquid. During the compression of such particles liquid is expressed initially from the interior of the particles (intraparticle volume) to the pores of the cake (extraparticle volume) and then from the extraparticle volume of the cake. The continuity

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[1] On the basis of an averaging volume V, the intrinsic i-phase average of a quantity ψ is identified to $\bar{\psi}_i^i = 1/V_i \int_{V_i} \psi_i \, dV$ and the phase average to $\bar{\psi}_i = 1/V \int_{V_i} \psi_i \, dV$. They are linked by the i-phase volume fraction within the averaging volume, ε_i.

equation for this case can be presented by the next expression:

$$\frac{\partial q_l}{\partial z} + \frac{\partial \varepsilon}{\partial t} - \Phi(P_2 - P_1) = 0 \tag{9}$$

where Φ is the empirical coefficient incorporating cake structure parameters, P_2 and P_1 are respectively the intraparticle and extraparticle pressure.

In other works the phenomenon of fine particles retention inducing the "clogging of the cake" has been incorporated. In this case the mass balance has been modified to take into account this flux of particles (49–54).

Momentum Balances

As opposed to the case of mass balance, momentum balance depends significantly on the approach used. Some debate still exists about the validity of certain equations proposed in the filtration literature. It would therefore appear important to us, to focus on these different approaches in order to analyse as objectively as possible the various results published in this field.

Momentum Balance on the Liquid Phase

In a general way the conservation of momentum, derived on the liquid phase, balances forces acting on the liquid, (force induced by interaction of the solid on the liquid, force induced by liquid pressure, gravity force, viscous force), convective and accumulation terms.

$$\Sigma F_l = \varepsilon \rho_l \left(\frac{\partial v_l}{\partial t} + v_{l,z} \frac{\partial v_l}{\partial z} \right) \tag{10}$$

The conventional approach derives the conservation of momentum equation, in one dimension, on a cake layer of dz thickness, for a Newtonian liquid. Only a few studies have been dedicated to the filtration of non-Newtonian liquids (55–58). Eq. (10) is usually directly presented under Darcy's Law form according to some classical assumptions: the process can be considered as being quasi-stationary, the convective term can be ignored, and the viscous and gravitational forces can be neglected (few works integrates the impact of gravitational forces within the framework of a conventional balance (59–61)).

$$\frac{\partial P_l}{\partial z} = -\frac{\mu_l}{k}(\varepsilon\ v_l) \tag{11}$$

with k being the permeability of the layer dz of the cake and P_1 the liquid pressure.

This expression of Darcy's law does not take into account the movement of solids within the cake. Therefore, in order not to neglect this aspect numerous researchers (19–20, 22, 24, 35, 58, 61–72) replace in the Eq. (11) the liquid velocity by the relative velocity of the mixture solid/liquid compared to that of the solid. Thus Darcy's law modified becomes:

$$\varepsilon(v_l - v_s) = -\frac{k}{\mu_l}\frac{\partial P_l}{\partial z} \tag{12}$$

This expression remains logical with the preceding relationship in the sense that if $v_s = 0$ one returns to the Eq. (11). It is also coherent with results obtained from the multiphase approach applying the volume averaging technique. Starting from the three-dimensional momentum balance written at the liquid phase scale:

$$\rho_l \frac{D\vec{v}_l}{Dt} = \nabla \cdot \underline{\underline{\sigma}}_l + \rho_l\, \vec{g} \tag{13}$$

where $\nabla \cdot \underline{\underline{\sigma}}_l$ represents the contribution of the surface forces and $\rho_l\ \vec{g}_l$ the one of the volume forces, it is possible to demonstrate, theoretically, Darcy's law in the case where the two phases are moving. The Equation (13) is then expressed in the following form:

$$\varepsilon\left(\overline{\vec{v}}_v^{\,l} - \overline{\vec{v}}_s^{\,s}\right) = -\frac{1}{\mu_l}\underline{\underline{k}} \cdot \nabla\left(\overline{P}_l^{\,l}\right) \tag{14}$$

with $\underline{\underline{k}}$ the permeability tensor.

The reader interested by this demonstration should consult Withaker's works (73, 74). Sfair (75) proposes a synthesis of these different works.

The comparison, in a one-dimension system, of Eqs. (12) and (14) (by identifying $\overline{P}_l^{\,l}$ to P_l) shows a good agreement between both ones and answers the question with regards to whether it is the intrinsic liquid-phase average pressure $\overline{P}_l^{\,l}$ or the average liquid pressure $\overline{P}_l$ that should be used in Darcy's law.

The modelling which refers to the works of Buscall and White (26) on sedimentation, derives the conservation of momentum equation on a volume of suspension at concentration ϕ assuming that inertial and bulk shear forces can be neglected (27, 29). It is usually written under the following form when gravitational force is not taken into account:

$$-\frac{\lambda\mu_1 a_p}{V_p}\phi r(\phi)(\vec{v}_l - \vec{v}_s) = \nabla P_l \tag{15}$$

The left hand term of Eq. (15) is hydrodynamic drag, exerted by the particles on the liquid. λ is the Stoke's drag coefficient, a_p is the particle size (which for spheres is the radius), and V_p is the particle volume. The function $r(\phi)$ is the hindered settling factor and accounts for the hydrodynamic interactions between the particles. Due to the difficulty to well-characterize usual (industrial) suspensions in terms of particle size, the hydrodynamic drag is generally not easy to derive. Consequently, another function denoted as $R(\phi)$, easier to measure than $r(\phi)$, is often used in Eq. (15), instead of $r(\phi)$ (31, 76)

$$R(\phi) = \frac{\lambda \mu_l a_p}{V_p} r(\phi) \tag{16}$$

$R(\phi)$can be related to the traditional permeability, $k(\phi)$ according to the following equation:

$$k(\phi) = \frac{\mu_l}{R(\phi)} \frac{1-\phi}{\phi} \tag{17}$$

Balance on the Solid Phase

In the conventional approach Tiller and Huang (16) referred to soil mechanics science and describe the filtration cakes as a stack of spherical particles surrounded by the liquid and supposedly only linked by point contacts. Using this basis and assuming that the process is quasi-stationary and that the inertial, gravitational, and bulk shear forces can be neglected the one-dimensional momentum balance, can be derived on a cake layer of thickness *dz*, by:

$$dF_s + \Omega_l \, dP_l = 0 \tag{18}$$

where F_s is the cumulative drag force and Ω_l the surface area allocated for the liquid.

By dividing Eq. (18) by the surface area of filtration (Ω) it comes:

$$\frac{dF_s}{\Omega} + \frac{\Omega_l}{\Omega} dP_l = 0 \tag{19}$$

At this point in their reasoning Tiller and Huang (16) use the points contact hypothesis to bring together the surface area allocated for the liquid and that for the filtration ($\Omega_l \# \Omega$). Then the integrated form of Eq. (19) becomes:

$$P_s + P_l = \Delta P \tag{20}$$

with $P_s (= \mathrm{F_s}/\Omega)$ the "fictive" pressure at the z-coordinate (which results from the forces exerted on the solid divided by the surface area of filtration) (16, 18, 67, 77), and ΔP the total difference in liquid pressure through the cake. This relationship is largely accepted and used throughout the literature in the following differential form:

$$\frac{\partial P_s}{\partial z} + \frac{\partial P_l}{\partial z} = 0 \tag{21}$$

In the approach based on Buscall and White works (26), the conservation of momentum equation on the solid was established similarly to its derivation on the liquid phase (27, 29):

$$-\frac{\lambda\,\mu_l\,a_p}{V_p}\,\phi r(\phi)\,(\vec{v}_s - \vec{v}_l) = \nabla P_s \tag{22}$$

In bringing together this equation with the one established on liquid it equally brought back to Eq. (21).

Abboud and Corapcioglu (78) and Zhao et al.'s (79, 80) approach was slightly different, whilst also referring to soil consolidation they preferred to use the general three-dimensional development proposed by Biot (81). This approach shows up the stress tensor of both the solid phase and the diphasic medium; this last factor being determined from rheological measurements carried out on the cake.

According to Biot (82), for an unstressed porous medium saturated in liquid, the stress tensor of the diphasic medium acts in accordance with the following equation:

$$\nabla \cdot \underline{\underline{\sigma_{diph}}} = 0 \tag{23}$$

with $\underline{\underline{\sigma_{diph}}}$ the tensor of the total stress applied on the cake supposed as an homogeneous diphasic material.

Within the framework of a conventional jacketed test (82), the stress tensor of the diphasic medium results from the contribution of the solid phase stress tensor (assumed to be isotropic) and the liquid phase stress tensor, by disregarding the effects of viscosity in the structure:

$$\underline{\underline{\sigma_{diph}}} = \underline{\underline{\sigma_s}} - P_l \underline{\underline{I}} \tag{24}$$

where $\underline{\underline{\sigma_s}}$ is the stress tensor of the solid phase and $\underline{\underline{I}}$ the unit tensor.

Reducing Eq. 23 to a one-dimensional system, allowed Zhao et al. (80) to observe that the stress applied to the diphasic medium did not depend on the z-coordinate and therefore to assimilate it to the pressure applied on the cake:

$$\sigma_{diph,zz} = -\Delta P \tag{25}$$

In this last equation, the subscript zz refers to zz-coordinate of the tensor, and the minus sign is drawn from the fact that the applied force is directed downwards whereas the axis direction is upwards. Based on Eqs. (23–25) Zhao et al. (79) defined P_s as the zz-coordinate of the solid phase stress tensor.

Although Eq. (21) is widely accepted and used throughout filtration literature, the validity and rigor of the definition of P_s remains strongly contested by the proponents of the multiphase approach (37, 38, 83, 84). Similarly to the derivation of momentum balance on the liquid, these authors applied the volume averaging technique on the conservation of momentum equation initially derived at the solid phase scale. Assuming usual assumptions which consider a quasi stationary change and neglect the inertial forces and the gravity impact, the averaged form of the conservation equation is:

$$\nabla \cdot \underline{\underline{\overline{\sigma}_s}} + \frac{1}{V}\int_{A_{ij}} \underline{\underline{\sigma_s}} \,.\, \vec{n}_{sl}\, dA = 0 \tag{26}$$

The integral term of Eq. (26) is the momentum loss from the solid phase to the liquid phase. In this term V is the averaging volume, $\vec{n}_{sl}$ the unit normal vector pointing from the solid to the liquid and A_{sl} the solid-liquid interfacial area.

By adding this equation with that established on the liquid phase and by putting down a certain number of hypotheses, in particular that the solid can be "conceptually envisioned as fluid continuum at the volume-averaged scale," it becomes (Appendix 1):

$$\nabla \cdot \underline{\underline{\overline{\tau}_s}} - \nabla \cdot (\overline{P}_l^l \underline{\underline{I}}) = 0 \tag{27}$$

with $\nabla \cdot \underline{\underline{\overline{\tau}_s}} - \nabla \cdot (\overline{P}_l^l \underline{\underline{I}}) = 0$ the shear stress tensor of the solid considered as a pseudo-fluid.

Consequently, from a one-dimensional form of Eq. (27) Tosun et al. (40) identified the conventional solid pressure (P_s) to the "surface contact force" in the solid phase $-\overline{\tau}_{s,zz}$. The physical signification of $\overline{\tau}_{s,zz}$ remains difficult to grasp, as it is difficult to accept a solid phase behaving as a fluid. Besides this definition of P_s is significantly different to that proposed by Willis et al. (37, 83) in earlier studies, according to which P_s could be assimilated to the zz-coordinate of the tensor of cumulative drag stress potential ($\underline{\underline{\overline{\sigma}}}^s_{drag}$) defined in (Appendix 2):

$$\nabla \underline{\underline{\overline{\sigma}}}^s_{drag} = -\frac{1}{V}\int_{A_{ls}} \underline{\underline{\tau_l}} \cdot \vec{n}_{ls}\, dA \tag{28}$$

The sum of the solid and liquid momentum balances can then be expressed in a one-dimensional form, as:

$$\frac{\partial \underline{\underline{\overline{\sigma}}}^s_{drag,zz}}{\partial z} + \varepsilon \frac{\partial \overline{P}_l^l}{\partial z} = 0 \tag{29}$$

So, according to Willis et al. (37, 83) the equation conventionally used Eq. (21), is only valid in the particular case where $\varepsilon = 1$ (which corresponds with the hypothesis of contact points formulated by Tiller and Huang (16).

Tien et al. in a series of articles (63, 64, 86) discussed the relevance of these two equations. They referred to the work done by Rietema (87), in order to explain that the different formulae which arise from different volume averaging methods and to propose two further equations capable of characterizing the merging of the two momentum balances.

$$\varepsilon\frac{\partial P_l}{\partial z}+\phi\frac{\partial P_s}{\partial z}=0 \tag{30}$$

$$\frac{\partial(\varepsilon P_l)}{\partial z}+\frac{\partial(\phi P_s)}{\partial z}=0 \tag{31}$$

In order to specify which equation should be retained, Bai and Tien (64) compared the results of simulations based on each of the four equations ((21), (29) with $P_s=\bar{\sigma}^s_{drag,zz}$, (30) and (31)), with filtration experiments carried out under constant pressure on suspensions of $CaCO_3$ and Kaolin. They concluded that the equation to be retained depended on the matter being tested and in particular on the "mechanism of the transfer of the drag force from particle to particle."

Bai and Tien's conclusion clearly highlighted the question of relevance concerning the models being used and implicitly those concerning the definition of P_s. This conclusion underlined the need for further work in this field.

CONSTITUTIVE EQUATIONS

Cake Compressibility

It is of little importance which approach is used in the handling of the momentum conservation of the solid phase which always involves the introduction of the solid pressure P_s in the compressibility of the cake equation. Even in situations where solid pressure is considered as one coordinate of the stress tensor, this equation must not be compared to a classical stress-strain relationship. The reliance on rheological strain introducing stress into the cake is generally not explicit. Nevertheless, the compressibility equation, as a relationship established from experiments, takes into account the true mechanical behavior of the system as it relates the force applied on the cake (which allows to blend the stress with the solid pressure) and the total strain (rheological strain and shrinkage associated to the displacement of the liquid phase). This strain can be, depending upon the approach taken, typified by different parameters attached unequivocally to volume fraction (of liquid or solid) or the void ratio ($e=\varepsilon/(1-\varepsilon)$).

The compressibility equation has been established by several studies on an experimental data base obtained by measurements carried out on a compression permeability cell (C-P Cell) (77, 88–92). This technique consists in placing the cake to be tested inside a hollow vertical cylinder sealed at its base with a filter medium. A moving porous piston upon which a mechanical force can be applied then imprisons the cake. According to the degree of force applied (assimilated to P_s) different points of balance are reached. These are characterized by the thickness of the cake (L) and each one is representative of a certain porosity (assumed to be homogenous through the cake) which can be calculated by:

$$\varepsilon_{av} = 1 - \frac{m_s}{\Omega\, L\, \rho_s} \tag{32}$$

where m_s is the mass of solid in the cell and Ω the surface area of filtration.

The majority of filtration models describe the mechanical change of the cake as a succession of states of equilibrium observed by measurements carried out on C-P Cell, and consider that the total strain depends solely on the applied force. The cake is compared to a homogenous medium where the rheological behavior is considered as being elastic. Proceeding on from this basis different laws have been proposed from tests carried out on a C-P Cell (Table 1). The most standard form is a power law which directly links ϕ to P_s.

$$\phi = \phi_0(a + bP_s)^n \tag{33}$$

where a, b, and n are fitting parameters and ϕ_0 is the solid volume fraction of the unstressed cake. By introducing a solid yield pressure ($P_{s,ref}$) into this type of equation, the situation where $\phi = 0$ when $P_s = 0$, which is physically inconceivable, can be avoided.

Although widely used the approach based on C-P Cell experiments, suffers from a number of shortcomings, the most important of which seems to be the friction on the walls which causes a loss of mechanical pressure and a non-uniformity of the cakes being tested (121–125). For complex systems such as biological sludge, a long time is needed to reach equilibrium states and biological activity also limits the use of the C-P cell. On the basis of the modelling which considers the filtration cake as a network possessing a compressive yield stress (P_y), several authors proposed some others experimental methods based on centrifugal or filtration tests (29, 31, 76). The centrifugation technique was applied to various materials such as, for example, polystyrene latex (126) and Alumina suspensions (108), filtration tests were essentially carried out on aluminia (127, 128) and Zirconia (129, 130) suspensions, and on potable water (128) and ferric and alum water treatment sludges (131). A method combining sedimentation and filtration tests was also used for both biological and non-biological wastewater sludge (43). In these methods it was considered that the compressive yield stress which characterises the material could be measured only when the medium is fully networked (above the gel point volume fraction ϕ_g), it was then directly related to the solid volume fraction. Among the different equations presented in the filtration literature (Table 1), the power law form (comparable to one referred as Eq. (33), is the most used:

Table 1. Compressibility equation depending on material nature

Mathematical relationships	Material
$\phi = \phi_0 (a + b P_s)^n$ $a = 1$ and $b = 1/P_{s,ref}$	Aluminium hydroxide (69), Attapulgite (67), Calcium carbonate (64, 86, 93–99), Calcium silicate (100), Carbonyl Iron (95, 96), Crushed dolomite (60), Flocculated Latex (101), Kaolin (20, 63, 71, 86, 95, 96, 99, 100, 102), Lignin sludge (100), Talc (102), Silica (20, 63, 103), Titania (99), Titania + perlite (77), Wastewater sludge (61, 62, 69, 70, 95, 96, 103, 104)
$a = 0$ and $b = 1$	Alumina (89), Calcium carbonate (15, 89), Carbonyl iron (89, 91), Cement material (16), Crushed dolomite (105), Kaolin (15, 16, 89, 91, 94, 106, 107), Kieselghur (16), Silica (89, 91), Talc (89, 91), Titania (89), Polystyrene latex (89, 91).
$\left.\begin{matrix} a = 1 \text{ and } b = 0 \quad \text{for} \quad P_s < P_{s,ref} \\ a = 0 \text{ and } b = 1 \quad \text{for} \quad Ps \geq P_{s,ref} \end{matrix}\right\}$	Cement material (16), Ferric Oxyde (58), Kaolin (77), Kieselghur (16), Titania + perlite (77).
$P_y = a[(\phi/\phi_g)^n - 1]$; for $\phi_g \leq \phi$	Alumina (108), Flocculated kaolin (109), Zirconia (110), Wastewater sludge (43, 111).
$P_y = a[\exp(b\phi) - \exp(c\phi)]$; for $\phi_g \leq \phi$	Flocculated kaolin (109).
$P_y = [1 - (\phi_g/\phi)]^{n_1} \exp(a\phi^{n_2} + b)$	Wastewater sludge (112).
$e = a - b \cdot \ln(P_s)$	Bentonite (113), Cement material (106, 107, 114), Ferric oxyde (115), Kaolin (116–118), Vegetal pulp (68, 119), Wastewater sludge (113).
$\dfrac{\partial \varepsilon}{\partial P_s} = a \,.\, \exp(b\,\varepsilon)$	Kaolin (120), Wastewater sludge (120).

$$P_y = a\left[\left(\frac{\phi}{\phi_g}\right)^n - 1\right] \tag{34}$$

where a and n are fitting parameters and ϕ_g is volume fraction at gel point (comparable to ϕ in Eq. (33)).

Other works, generally more specifically focused on cake expression operation, the compressibility equation was derived on the basis of a differential form of Hooke's law:

$$\frac{\partial \sigma}{\partial \in} = E \tag{35}$$

where σ is the total stress applied on the cake and $\in$ is the cake strain. By mechanical analogy the cake is assimilated to a spring characterized by a compressibility modulus E which represents the reaction of the medium to a mechanical action. Thus, it is influenced by the change in the internal structure of the cake. Several authors (68, 106, 119, 132) basing their work on that done on soil mechanics (amongst others (133)), characterized this change by an expression in the following form:

$$E = b_E\ P_s = -(1+e)\frac{\partial P_s}{\partial e} \tag{36}$$

with b_E a constant. Eq. (36) can be, after linearization and integration written as:

$$e = a - b\ln(P_s) \tag{37}$$

with a and b some fitting parameters.

However, this last equation is no longer valid when P_s tends to 0. Murase et al. (117) proposed to consider a yield stress $P_{s,ref}$ under which the void ratio (e) takes constant minimal value.

Likewise, using the results from soil mechanics, Shirato et al. (134) suggested to take into account time dependence, by comparing the cake to a visco-elastic medium represented by a Kelvin-Voigt system, which, in the context of a mechanical analogy associated in parallel a spring and a dashpot (Table 2). The effect of time can then be represented by the dashpot, as:

$$\sigma = G\frac{\partial \in}{\partial t} \tag{38}$$

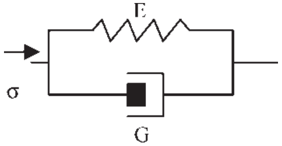
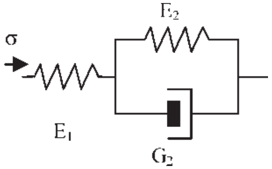
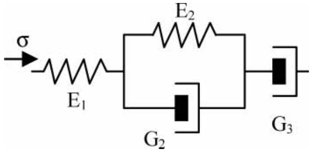
with G the viscosity modulus.

The change of void radio over time can be obtained by applying Boltzmann's superposition principle. Therefore, after integration, by considering $a_c(=(1+e)/E)$ to be constant (134) and in assimilating the stress to the difference $P_s - P_{s,in}$, where $P_{s,in}$ represents the solid pressure at the beginning of compression, it becomes:

$$\frac{\partial e}{\partial t} = -a_c\frac{E}{G}\frac{\partial}{\partial t}\int_0^t (P_s - P_{s,in})\exp\left(-\frac{E}{G}(t-\tau)\right)\partial\tau \tag{39}$$

By representing cake compression as a combination of a spring and a Kelvin-Voigt modules in series, Shirato et al.'s (134–136) aim was to present a more accurate model to study cake behavior under compression (Table 2). A dashpot was added by Lee et al. (137–139) to complete the model (Table 2) in order to typify, in the case of waste sludges, erosion of the bound water. However, this assumed that the bound water could be removed by mechanical means, which remains a question still open to debate (142).

Table 2. Rheological models based on mechanical analogy

Mechanical analogy	Rheological model	Material
	$\frac{\partial e}{\partial t} = -a_c \frac{E}{G}\frac{\partial}{\partial t}\int_0^t (P_s - P_{s,in}) \exp\left(-\frac{E}{G}(t-\tau)\right)\partial\tau$	(132)
	$\frac{\partial e}{\partial t} = -a_c \frac{E_2}{G_2}\frac{\partial}{\partial t}\int_0^t (P_s - P_{s,in}) \exp\left(-\frac{E_2}{G_2}(t-\tau)\right)\partial\tau - a_E \frac{\partial P_s}{\partial t}$	Clay (106, 135–138, 140, 141), Focculated bentonite (113), Kaolin (106, 116, 137, 140, 141), Solka-floc (141), Vegetal pulp (116).
	$\frac{\partial e}{\partial t} = -a_c \frac{E_2}{G_2}\frac{\partial}{\partial t}\int_0^t (P_s - P_{si}) \exp\left(-\frac{E_2}{G_2}(t-\tau)\right)\partial\tau - a_E \frac{\partial P_s}{\partial t} - a_c \int_0^t P_s \, d\tau + a_G P_{si}$	Wastewater sludge (137–139)

Lanoisellé et al. (48) used four Kelvin-Voigt elements in series to describe the creep behavior of biological materials like oilseeds. In their model the Kelvin-Voigt elements physically represent the four consolidation steps: the primary and creep consolidation of the extraparticle volume, and the consolidation of both extracellular and intracellular volumes.

Within these different models the parameters $a_E = (1 + e/E_1)$, $a_C = (1 + e/E_2)$, $a_G = (1 + e/G_3)$ and E_2/G_2 (Table 2) are assumed to be constant and therefore independent of the cake compactness. La Heij et al. (61, 62) studied the case where the parameters change with void ratio; empirical relationships were used to characterise these variations. Kamst (65) investigated, from Boltzmann's superposition principle, the visco-elastic case of palm-oil filter cake for which the dependence of the ratio strain/pressure with time change over time. He used an empirical form of the compressibility equation for which the void ratio depends on both the solid pressure and time. In both studies, the results obtained on highly compressible material such as wastewater sludges and palm-oil filter cake are encouraging, but the problems related to defining the different fitting parameters remain.

All the basic equations presented up to now have used the solid pressure notion, and their validity depends upon the aspects evoked in the previous paragraph. Smiles (32, 34) and Smiles and Kirby (33) got round this problem by introducing "the piezometric potential" of liquid which can be directly measured by using a piezometric tube. The empirical relationship was thus obtained between the rate volume and this potential and the following relationship was proposed:

$$e = e_1 log(-\psi) + e_2 \qquad (40)$$

where e_1 and e_2 are fitting parameters.

In order to use this relationship, Darcy's law has to be modified by replacing P_l with ψ.

Permeabililty Equation

In the case of fluid mechanics in porous mediums it is standard practice to use Kozeny-Carman's law to evaluate the permeability of the medium in function of its porosity. It was essentially used in previous works carried out in filtration field (7, 15, 88–90) and gave some relatively good results, mainly with mineral material. However, depending on the material nature, this law does show its limits when considering cake filtration (91) and other (empirical) expressions have been introduced (Table 3). In the conventional approach, they define the permeability (or the specific resistance of filtration) as a function of the porosity or of the solid pressure (knowing that in many studies ε, or ϕ, have been shown to be directly related to P_s). The most standard form is a power law similar to

Table 3. Constitutive equations in terms of cake permeability (or equivalents)

	Mathematical relationships for filter cake permeability (and equivalents)^a	References
As a function of ε or ϕ	$\alpha_m = \frac{k_K S_0^2}{\rho_s}\frac{(1-\varepsilon)}{\varepsilon^3}$; with k_k the Kozeny constant and S_0 the specific surface of particle	Alumina (89), Aluminium hydroxyde (7), Calcium carbonate (89), Carbonyl Iron (89), Clay (143), Kaolin (89, 143), Kieselghur (7), Polystyrene latex (89), Silica (89), Solka-floc (89), Talc (89), Titania (89).
	$\alpha_m = \frac{a'}{\rho_s}\varepsilon^{n'}$	Alumina (91), Calcium carbonate (91), Carbonyl iron (91), Kaolin (91), Silica (91), Talc (91), Titania (91).
	$k = \mathrm{a}'\exp[\mathrm{b}'(\varepsilon - \varepsilon_0)]$	Vegetal pulp (65).
	$k = \mathrm{a}'\exp(\mathrm{b}'\,\varepsilon)$	Kaolin (120), Wastewater sludge (120).
	$R = \mathrm{a}'\phi^{n'} + \mathrm{b}'$	Wastewater sludge (112).
	$R = \mathrm{a}'(1-\phi)^{n'}$	Kaolin (109), Wastewater (43, 111).
As a function of Ps	$\alpha_m = \alpha_{m,0}\,(a' + b'\,P_s)^{n'}$ with: $a' = 0$	Calcium carbonate (18), Cement material (16, 21), Crushed dolomite (107), Kaolin (16, 18, 21, 118), Kieselghur (16), Polystyrene Latex (18), Talc (18).

$\left.\begin{array}{l} a' = 1 \textit{ and } b' = 0 \textit{ for } P_s < P_{s,ref} \\ a' = 0 \textit{ and } b' = 1 \textit{ for } P_s \geq P_{s,ref} \end{array}\right\}$	Cement material (16), Kaolin (77), Kieselghur (16), Titania + perlite (77).
$a' = 1$ and $b' = 1/P_{s,ref}$	Aluminium hydroxyde (69), Attapulgite (67), Calcium Carbonate (86, 93, 94, 99), Calcium silicate (100), Clay (106, 114), Cement material (106, 114), Flocculated Latex (101), Kaolin (20, 63, 71, 77, 86, 99, 100, 102, 106, 114, 121), Lignin sludge (100), Silica (20, 63), Talc (102), Titania (99), Titania + perlite (77), Wastewater sludge (69, 144).
$\alpha_v = \alpha_{v,0}(a' + b'\ P_s)^{n'}$ with $a' = 0$	Vegetal pulp (119).
$a'=1$ and $b' = 1/P_{s,ref}$	Carbonyl Iron (95), Kaolin (95, 96, 103), Silica (20, 63, 103), Wastewater sludge (95, 96, 103, 104).
$k = k_0\ (a' + b'\ P_s)^{n'}$ with $a' = 1$ and $b' = 1/P_{s,ref}$	Calcium Carbonate (64, 97), Kaolin (64), Wastewater sludge (61, 62, 70).

[a] Equivalents are the specific resistance of filtration and the hindered settling factor.

the one proposed for compressibility equation.

$$k = k_0\,(a' + b'\,P_s)^{n'} \tag{41}$$

where a', b', and n' are fitting parameters and k_0 is the permeability of the unstressed cake. These parameters are usually estimated from permeability tests carried out on a Filtration Permeability Cell, by determining the flow rate induced by a constant load (constant head reservoir) through the confined cake. Consequently the confidence which one can have in that kind of data widely depends on the confidence which one can have in the results issued from C-P Cell experiments, as it was discussed previously (see Section 4.1). As alternative methods Murase et al. (117, 118) and Teoh et al. (99) determined cake properties from global cake filtration data whereas Johansson et al. (100) based their determination on internal filter cake measurements, such as pressure and porosity distribution. Very recently, Teoh et al. (99) compared C-P cell and cake filtration data and showed a good agreement between both techniques depending on the way P_s and P_1 are related (see section 3.2.2 especially Eqs. (21), (29) with $P_s = \overline{\sigma}^s_{drag,zz}$, (30) and (31)).

The specific resistance of filtration is, depending upon the authors, either expressed in volume or in mass:

$$\alpha_m\,k\,\rho_s = \alpha_v\,k = 1 \tag{42}$$

The authors referred to dry cake often use the ratio of mass (or volume) of solids deposited by volume of filtrate collected instead of the ratio of mass (or volume) of the wet cake by volume of the filtrate collected. Consequently, Eq. (42) has to be modified, and resistances relative to deposited solids ($\alpha_{m,s}$ and $\alpha_{v,s}$) have to be introduced:

$$\alpha_{m,s}\,k\,\phi\,\rho_s = \alpha_{v,s}\,k\,\phi = 1 \tag{43}$$

In the approach based on compressive rheology concept, the permeability equation was generally derived from filtration tests, in terms of hindered settling factor. The most commonly utilized functional form is given by:

$$R(\phi) = a'(1-\phi)^{n'} \tag{44}$$

with a' and n' fitting parameters.

The correlation between this parameter and the conventional permeability (or specific resistance of filtration) was discussed previously (see Section 3.2.1).

To explain the fact that the overall dewatering process depends at the same time on the cake compressibility and its permeability numerous authors introduced a filtration diffusivity. This notion is also related to the possibility to derive the filtration problem in Lagrangian coordinates (rather than Eulerian coordinates). In this case, based on the fact that cake thickness changes over time, a material coordinate ω (representing the volume, per filtration surface area, of solid deposited into a z cake thickness) is defined so that:

$$\left(\frac{\partial}{\partial t}\right)_{\omega} = \left(\frac{\partial}{\partial t}\right)_{z} + \frac{q_s}{\varepsilon_s}\left(\frac{\partial}{\partial z}\right)_{t} \tag{45}$$

Using this change of coordinates, the filtration problem can be derived in a diffusion-like equation (14, 36).

$$\frac{\partial e}{\partial t} = -\frac{\partial}{\partial \omega}\left(D(e)\frac{\partial e}{\partial \omega}\right) \tag{46}$$

The diffusive formulation was utilized by a number of authors (32–36, 145–147). D is then generally derived by adjustments to cake filtration data. Based on compressive rheology approach the following relationship is usually considered (43, 110, 111, 130):

$$D(\phi) = \frac{dP_y}{d\phi}\frac{(1-\phi)^2}{R(\phi)} \tag{47}$$

In order to reconcile the different modelling approaches, it can be noted that the cake diffusivity is directly related to the modified consolidation coefficient (commonly referred as C_e) defined in soil mechanic theory and often used in works relative to expression stage. In a number of these works C_e was supposed to be constant (72, 107, 114, 140).

The standard relationships used to characterise cake permeability (and presented above) show a monotonically increasing function of the volume fractions (ε or ϕ) or of the solid pressure. But Willis and Tosun (37, 86) called into question this particular feature by showing experimentally the presence of a minimum. According to our knowledge, this discussion was not reconsidered by other authors and remains opened to debate.

Modelling of Cake Properties on Particle-Particle Interaction Basis

To limit the use of empirical equations, some recent studies have attempted to develop "physical models" to predict cake properties based on consideration of the interactions between particles. Bowen and Jenner (148) assumed that the disjoining pressure, which is the sum of van der Waals, the electrostatic double layer, and hydration forces, is equal to the solid pressure as defined by Eqs. (18–20). Filter cake porosity is computed by a simple cell model based on close packing of the particles and a Kozeny-Carman equation was used to calculate the local specific resistance. In the same way, Kapur (149) computed compressive yield stress P_y based on a model including these typical interactions between particles (150, 151). Unfortunately, the developed models are, in their current form, adapted only to very specific (academic) situations. More recently Dustan et al. (152) tried to use such an approach to more real system (non-rigid, non-spherical, heterodisperse, flocculated system). Their efforts have finally yielded semi-empirical relationships with fitting parameters. Consequently, in this work, the interest of theoretical development was essentially to guide the form of empirical constitutive equations and reduce the number of required undetermined parameters.

Koenders and Wakeman (153–155) proposed an alternative model which assumes that solid pressure is due to the double layer force only. The approach yielded significant results, albeit it was also validated only for limiting systems.

Although, the attractiveness of a priori computation of filter cake properties is clear, the research in this field is still in early stages of development. Further work is really needed to yield result useful at engineering scale.

MODELS APPLICATION

Resolving the equations of the models is needed when moving onto the stage of simulation. The complexity of the methods to be implemented will be directly linked to the number of space dimensions required according to the system geometry and to the medium anisotropy. Most of the published works are devoted to cylindrical Filtration Compression Cells and only few studies deals with industrial devices. They generally simulate the change of volume of filtrate, cake thickness, and pressure and porosity distribution through the cake over filtration and/or expression stages. In a usual way, the isotropy of the cake (according to the thickness) was assumed and a one-dimensional system is considered.

Most of the filtration models were successfully used for a number of materials. Numerical solving was commonly used for the simulation step (20, 24, 27, 35, 43, 61, 65). However, it is really difficult to compare the different results to each other due to the large number of operating conditions investigated.

The applications to the full dewatering process (filtration and expression stages) generally required some adaptations especially in terms of rheological modelling (see Section 4.1). Using the conventional approach as a starting point, Shirato et al. (106, 114, 132, 134, 141) widely investigated filter cakes expression, and proposed an analytical analysis to describe this particular stage. The Shirato et al.'s results were essentially validated on clay suspensions, but other materials such as solka-floc (141), vegetal pulp (116), or gel (156) showed good agreement with the model developed by these authors.

Nevertheless, few authors pointed out the limits of currents models (of filtration and of expression) to accurately describe the dewatering of complex material such as pulps or biological sludges (13, 14). Although some interesting results on wastewater biological sludge have been reported by both the compressive rheology (43) and the conventional approaches (24) (models based on multiphase theory and on Smiles' work (32) were not tested on such materials) some questions remain about this kind of material. The sludge cake usually exhibits visco-elastic behavior and the use of elastic correlations, such as classical compressibility equations, is not well-adapted. Moreover, few authors (69, 157) have noted, for such material, the formation of a highly compacted (and then highly resistive) skin at the cake/medium interface which renders as well questionable the use of classical constitutive equations. The difficulty to carry out reproducible and reliable measurements on wastewater sludge is also a problem for its dewatering modelling (14). Besides, the basic assumption which considers, in the case of the conventional approach, that only contact point exists among particles, does not hold any more for such compressible materials, and requires revision of the theory.

In a series of papers, Tiller and co-workers have tried to better investigate the particular issue of highly compressible materials, such as wastewater sludge, based on the conventional approach. They notably showed that there is negligible effect of pressure (beyond a certain value which was generally low) on both the filtrate flow rate and the average solid fraction of filter cake (95, 96, 104). They validated this result by experiments and referred to the previously mentioned skin phenomena at the cake/medium interface to explain this unexpected behaviour. Although these are interesting results, the conventional theory is not well-adapted to highly compressible materials, as it generally considers that the solid velocity in the filter cake is constant. Tiller et al. (103) and Lee et al. (22) discussed this feature in a somewhat controversial way, and concluded that the error incurred when estimating cake characteristics for a null solid velocity is acceptable regardless of the model simplicity. According to Tiller et al. (103), more complicated models contain a large number of parameters which can be difficult to measure, then the question of which model leads to the more accurate results remains.

Concerning dewatering of vegetal pulps the works of Lanoisellé et al. (48, 119) and Kamst (65) are an interesting basis for further studies in this field.

Although it is of engineering interest, only few studies tried to scale-up the benefit of dewatering modelling to industrial devices. Recently, Stickland et al. (131) have detailed the theoretical development of a model of a fixed-cavity plate-and-frame filter press enable to predict non biological wastewater sludge dewatering. It is based on the compressive rheology concept and simplifies the process to one-dimension. To take into consideration the fact that the slurry is fed at the center of the plate rather than at a plane, and then simulate local radial concentration gradients, Wells (158) proposed a two-dimensional model, but without experimental validation. Wells and Sivakumaran (159) also considered a similar two-dimensional approach to analyze the belt filter press, but no simulations were provided in this work. In fact, the development of two- or three-dimensional models well adapted to industrial devices requires to consider shear phenomena in the modelling.

CONCLUSION

This work has provided an overview of the main aspects of transport phenomena modelling in cake filtration and expression processes. Most studies combined fundamental equations (continuity equation and momentum balance) and material properties parameters (compressibility and permeability). A comparative analysis of the different models and approaches has been proposed, underlining their particular analogies and differences. Several points are still open to debate, notably the derivation of the momentum balance of the solid phase and the determination of the constitutive equation from experimental data. The application of the current models to complex materials such as sludge and pulp is not obvious, because of their usual high compressibility and of the lack of reliable and reproducible experimental data, and a significant amount of effort is needed in this field. This work has highlighted that most of the modelling works were committed to an analysis in one dimension of the Filtration Compression Cell system. Further research is required to arrive at multidimensional models, in order to well simulate the behavior of industrial devices.

MATHEMATICAL NOTATIONS

$\vec{x}$	vector
$\underline{\underline{x}}$	tensor
$\nabla\cdot$	divergence operator
∇	gradient operator
D/Dt	particulate derivative

APPENDIX

Conservation of Momentum Equation

From volume averaging, the momentum conservation equation can be derived under the following form (for $i = l$ or s):

$$\overline{\rho_i \frac{D\vec{v}_i}{Dt}} = \overline{\rho_i \vec{g} + \nabla \cdot \underline{\underline{\sigma_i}}} \tag{A1}$$

with $\underline{\underline{\sigma_i}}$ the stress tensor, ρ_i the density, and $\vec{v}_i$ the velocity vector of the i-phase.

Using Gray decomposition, transport, and divergence theorems and accepting the usual assumptions specific to volume averaging such as incompressibility of each phase, the gravitational acceleration constant, and non-adherence between the two phases (41, 42, 75), Eq. (A1) becomes:

$$\overline{\rho_i\vec{g}+\nabla\cdot\underline{\underline{\sigma_i}}}=\overline{\rho}_i\vec{g}+\nabla\cdot\overline{\underline{\underline{\sigma_i}}}+\frac{1}{V}\int_{A_{ij}}\underline{\underline{\sigma_i}}\cdot\vec{n}_{ij}dA \tag{A2}$$

where the integral term represents the momentum loss from the i-phase to the j-phase. In this term V is the averaging volume, $\vec{n}_{ij}$ the unit normal vector pointing from i-phase, to j-phase, and A_{ij} the interfacial area between i-phase and j-phase.

From classical filtration assumptions which consider a quasi-stationary change and neglect the inertial term and the gravity impact, eq. (A2) becomes:

$$\nabla\cdot\overline{\underline{\underline{\sigma_i}}}+\frac{1}{V}\int_{A_{ij}}\underline{\underline{\sigma_i\cdot\vec{n}_{ij}}}\,dA=0 \tag{A3}$$

The stress tensor of a Newtonian liquid phase is given by:

$$\underline{\underline{\sigma_l}}=-P_l\,\underline{\underline{I}}+\underline{\underline{\tau_l}} \tag{A4}$$

with $\underline{\underline{\tau_l}}$; the liquid shear stress tensor.

Using Eq. (A3) on both the solid and the liquid phases, it comes, after addition:

$$\nabla\cdot\overline{\underline{\underline{\sigma_s}}}-\nabla\cdot(\overline{P}_l\,\underline{\underline{I}})+\nabla\cdot\overline{\underline{\underline{\tau_l}}}+\frac{1}{V}\int_{A_{ls}}(\underline{\underline{\sigma_s}}\cdot\vec{n}_{sl}+\underline{\underline{\sigma_l}}\cdot\vec{n}_{ls})\,dA=0 \tag{A5}$$

Then, assuming the stress continuity and neglecting shear stresses, equation (A5) becomes:

$$\nabla\cdot\overline{\underline{\underline{\sigma_s}}}-\nabla\cdot(\varepsilon\,\overline{P}_l^l\,\underline{\underline{I}})=0 \tag*{(A5)}$$

Tosun et al. (40) assume (debatable assumption) that the solid phase can be "conceptually envisioned as the fluid continuum at the volume-averaged scale." Consequently, they consider that:

$$\overline{\underline{\underline{\sigma_s}}}=-\phi\overline{P}_s^s\underline{\underline{I}}+\overline{\underline{\underline{\tau_s}}} \tag{A6}$$

Furthermore, they admit (other debatable assumption) that if the surface tension forces can be neglected the pressure around the interfacial area between the liquid and solid phase (A_{ls}) is continuous and can be given by:

$$\overline{P}_l^l=\overline{P}_s^s \tag{A7}$$

From equations (A6) and (A7), remembering that $\varepsilon+\phi=1$, equation (A5) can be written as:

$$\nabla\cdot\overline{\underline{\underline{\tau_s}}}-\nabla\cdot(\overline{P}_l^l\,\underline{\underline{I}})=0 \tag{A8}$$

which is Eq. 27 in the main text.

APPENDIX 2

Definition of P_s from Volume Averaging on the Liquid Phase

Neglecting viscous forces within the liquid phase (but not at the solid-liquid interface), the simplified equation of momentum conservation of the liquid, can be written according to equation (A3), under the following form:

$$\nabla \cdot (\overline{P_l \underline{\underline{I}}}) + \frac{1}{V}\int_{A_{ij}} (P_l \underline{\underline{I}}) \cdot \vec{n}_{ls}\, dA - \frac{1}{V}\int_{A_{ij}} \underline{\underline{\tau_l}} \cdot \vec{n}_{ls}\, dA = 0 \qquad \text{(A9)}$$

Neglecting the dispersions on P_1 and noting that $1/V \int_{A_{ij}} (P_l \underline{\underline{I}}) \cdot \vec{n}_{ls}\, dA \approx -\overline{P}_l^l \underline{\underline{I}}\, \nabla\varepsilon$ (41), it comes:

$$\varepsilon \nabla \cdot (\overline{P}_l^l \underline{\underline{I}}) - \frac{1}{V}\int_{A_{ij}} \underline{\underline{\tau_l}} \cdot \vec{n}_{ls}\, dA = 0 \qquad \text{(A10)}$$

Then, Willis and Tosun (37) introduce a scalar potential $\bar{\sigma}^s_{drag,zz}$ such as:

$$\nabla \cdot \underline{\underline{\overline{\sigma}}}^s_{drag} = -\frac{1}{V}\int_{A_{ij}} \underline{\underline{\tau_l}} \cdot \vec{n}_{ls}\, dA \qquad \text{(A11)}$$

This potential corresponds, according to them, to the cumulative drag stress; its coordinate $\bar{\sigma}^s_{drag,zz}$ can, in the particular case where $\varepsilon = 1$, be assimilated to the conventional solid pressure (P_s). To justify their approach Willis and Tosun (37), refer to some results presented by Willis et al. (83) which show, from calculations and experiments, that into a filter chamber the cumulative drag stress at the filter medium level (σ_0) can be derived as:

$$\overline{\sigma}^s_{drag,o} = \varepsilon_{av}\, \Delta P \qquad \text{(A12)}$$

with ε_{av} the average porosity across the cake.

This result can also be established, using equation (A11), by the integration of equation (A10) over a filtration cake (from $\bar{\sigma}^s_{drag,0}$ to $\bar{\sigma}^s_{drag,zz} = 0$ at the top of the cake) in which the gradient of porosity can be ignored (i.e. a cake with a constant porosity).

Tosun and Willis (85) also tried to show the full justification of their approach by deriving for three particular situations (tube, annulus, and concentric spheres) equation (A10), from conventional pressure drop relationships (of the Poiseuille-Hagen law type).

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